The Persoz's gephyroidal model described by a maximal monotone differential inclusion

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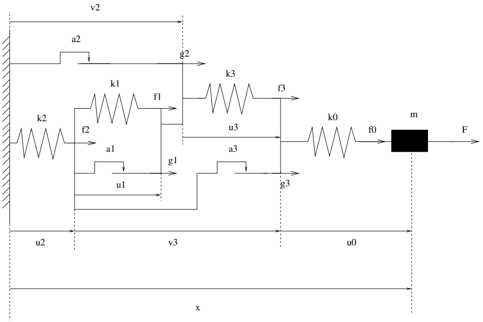
Plan

Introduction

The rheological Persoz's gephyroidal model, made out of some elementary rheological models (dry friction element and linear spring) can be covered by the existence and uniqueness theory for maximal monotone operators. Moreover, classical results of numerical analysis allow to use a numerical implicit Euler scheme, with order of convergence one. Some numerical simulations are presented.

The rheological Persoz's gephyroidal model



We consider the model involving

1. four springs with stiffness k_0, k_1, k_2 and k_3
2. three St-Venant elements with threshold α_1, α_2 and α_3
3. one material point of mass m

Main idea

The rheological Persoz's gephyroidal model is governed by differential inclusion of the form:

$$\begin{cases} \dot{X}(t) + MA(X(t)) \ni G(t, X(t)), \text{a.e. on }]0, T[, \\ X(0) = X_0, \end{cases}$$

where

- M is a invertible matrix
- X is a function from $[0, T]$ in $\mathbb{R}^p$
- A is a maximal monotone graph on $\mathbb{R}^p$
- G a function from $[0, T] \times \mathbb{R}^p$ in $\mathbb{R}^p$

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Notations

We introduce the classical following notations (e.g. as in [BSL00] :

- for $i \in \{0, ..., 3\}$, $k_i \longrightarrow$
  - displacements u_i
  - forces f_i
- for $i \in \{2, 3\}$, $\alpha_i \longrightarrow$
  - displacements v_i
  - forces g_i
- k_1 and $\alpha_1 \longrightarrow$
  - displacements $v_1 = u_1$
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- Let x be the abscissa of material point with mass m, and F be the external force, applied to this point.

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Maximal monotone used graph

We consider σ and β the multivalued maximal monotone graphs defined by

$$\sigma(x)=\begin{cases}-1 & \text{if } x<0,\\ 1 & \text{if } x>0,\\ [-1,1] & \text{if } x=0.\end{cases}\quad \beta(x)=\begin{cases}\emptyset & \text{if } x\in]-\infty,-1[\cup]1,+\infty[\\ \{0\} & \text{if } x\in]-1,1[,\\ \mathbb{R}_- & \text{if } x=-1,\\ \mathbb{R}_+ & \text{if } x=1.\end{cases}$$

According to [Bre73], these graphs are maximal monotone and

$$\sigma=\partial|.|,\quad \beta=\partial\psi_{[-1,1]},\quad \sigma=\beta^{-1}.$$

The differential inclusion governing the model

After computation, we obtain

$$\dot{X}(t) + M\partial\psi_{\mathcal{C}}(X(t)) \ni G\big(t, X(t)\big),$$

where

- X is a function from $[0, T]$ in $\mathbb{R}^5$
- M is a symmetric positive definite matrix (under some assumptions)
- $\partial\psi_{\mathcal{C}}$ is the subdifferential of the indicatrix of a closed convex of $\mathbb{R}^5$ for the scalar product defined by

$$< X, Y >_M = X^T M^{-1} Y,$$

- G is a regular function from $[0, T] \times \mathbb{R}^5$ in $\mathbb{R}^5$

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Existence and uniqueness

Theorem (Existence and uniqueness)

Let $(\alpha_i)_{1\leq i\leq 3}$ be positive numbers, $(k_i)_{0\leq i\leq 3}$ be positive numbers satisfying

1. $k_0 = 0$ *and for all* $i \in \{1,2,3\}$, $k_i > 0$
2. *or* $k_0 > 0$ *and at least two numbers among* k_1, k_2 *and* k_3 *are non negative.*

There is a unique solution X *in* $W^{1,\infty}(0, T; \mathbb{R}^5)$ *for the previous differential inclusion.*

Main idea of the proof

The proof of this result is based on the following idea : if $\mathbb{R}^5$ is equipped with its canonical scalar product, and with another scalar product

$$< X, Y >_M = X^T M^{-1} Y,$$

where M is symmetric positive definite, then we can relate the sub-differential $\partial\phi$ of ϕ relatively to the canonical scalar product and the sub-differential $\partial_M\phi$ relatively to $<,>_M$ by

$$\partial_M\phi(X) = M\partial\phi(X).$$

We apply then results proved in [BS00, BS02].

Numerical scheme

Theorem

Let N be an integer, $h = T/N$, $h_p = hp$ and X^p defined by

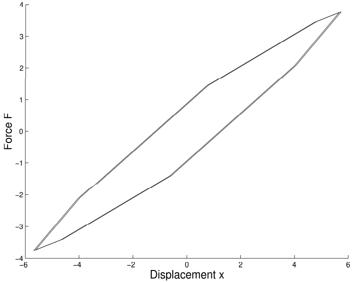
$$X^{p+1} = proj_{\mathcal{C},M^{-1}}\left(X^p + G(t_p, X^p)\right),$$

where $proj_{\mathcal{C},M^{-1}}$ is the orthogonal projection on the convex $\mathcal{C}$ for the previously defined norm on $\mathbb{R}^5$. Denote $X_h \in C^0\left([0,T];\mathbb{R}^5\right)$ the linear interpolation at time $t_p = hp$ of the solution X^p. The numerical scheme is of first order.

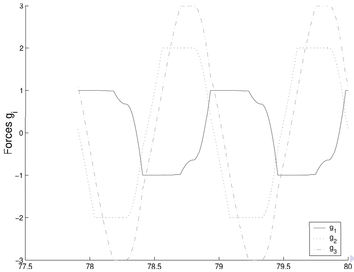
Quasistatic problems

In the quasistatic case, the mass m can be equal to zero.
Existence, uniqueness and numerical scheme hold.

Numerical simulations : hysteresis cycle



Numerical simulations : Specificity of gephyroidal studied model



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