A driven and braked wheel described by differential inclusion

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Summary. A wheel submitted to the friction forces exerted by the ground and by a brake system is studied. By using multivalued operators, we can write the constitutive laws of the wheel as a differential inclusion. We can connect a chassis to one, two or four of these wheels, by obtaining a differential inclusion of the same kind as the previous one. More generally, many applications can be offered in the field of nonlinear dynamics of wheeled vehicles.

Introduction

For more details, the reader is referred to [1, 2].
This communication is devoted to the study of a simple wheel, subjected to a motor torque and two friction forces: one exerted by the ground and one by a brake system. Some authors have already studied the complete mechanical behavior of vehicle in non linear dynamics, for example in [3, 4, 5, 6, 7, 8] or [9] for continuous modelization of brake, or in the numerous works of Hans True *et al.* in [10, 11, 12, 13, 14, 15, 16, 17]. This communication does not give as complete a description as that found in these references. We focus instead on the following point: we try to show here that the maximal monotone formalism used in [18, 19, 20, 21, 22, 23, 24] and applied to some elastoplastic models is well-adapted to the description of the dynamical behavior of the wheel. This allows the differential equations to be correctly written, and results concerning the existence, uniqueness and the convergence of the numerical scheme to be obtained. Indeed, using multivalued operators and differential inclusions, we describe both kinds of behavior (static and dynamic) of Coulomb's friction law. Moreover, the studied model is more complete than that in [6], where the case in which the wheel is locked by the brake system is not taken into account.
We first present the friction laws are presented in the simpler case of Coulomb's law. Then, we study a very simple model: wheels-chassis-ground. Theoretical existence and uniqueness and results are given, as well as a numerical scheme with some simulations. Finally, some generalizations are proposed.

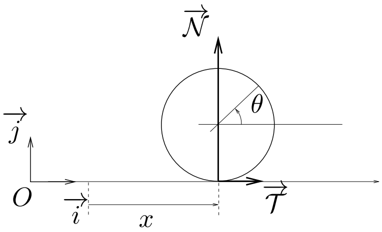
Presentation of the studied wheel and of the two friction laws

Coulomb's law

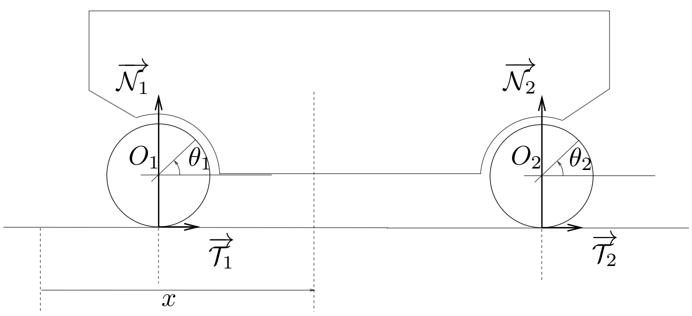
We recall Coulomb's friction law: consider the action $\overrightarrow{\mathcal{R}}$ of a solid on an another solid. This action can be decomposed into a normal component $\overrightarrow{\mathcal{N}}$, perpendicular to the tangential surface of contact between the solids, and a tangential component $\overrightarrow{\mathcal{T}}$. As long as the ratio $\mathcal{T}/\mathcal{N}$ does not exceed a certain limit μ_f, there is adherence and the solid remains at rest. Once this value is reached, there is slippage, we have $\mathcal{T}/\mathcal{N} = \mu_f$ and the tangential force is opposed to the relative velocity between the two solids. Some expressions of μ_f are possible, for example according to the relative velocity. See for example [3, 6].
We assume here that the dynamic friction coefficient (in slip or dynamic phase) is equal to the static friction coefficient (in adherence or static phase), and that this coefficient is constant and uniform, as in [25], [5, chapter 5] or [6, Fig. 2.3c) p. 85].

Friction laws of the wheel

We present the study of a simple wheel, subjected to a motor torque and two friction forces: the first one is exerted by the ground and the second one by a brake system (see [1, 2]). These two forces are governed by the Coulomb's law. We assume that the ground is plane and horizontal and that the wheel moves in a plane. The wheel is defined by two parameters: x, the abscissa of its center of gravity, and θ, the angle of rotation of the wheel (relative to a fixed direction) (see Fig. 1(a)). Let $\left(0, \overrightarrow{i}, \overrightarrow{j}\right)$ be a reference frame. We assume that the wheel is in contact with the plane ground. The



(a) A simply braked wheel



(b) A more complete model "two wheels-chassis-ground"

Figure 1: Two models with braked wheels

ground action on the wheel is denoted by $\overrightarrow{\mathcal{R}} = \overrightarrow{\mathcal{T}} + \overrightarrow{\mathcal{N}} = \mathcal{T}\overrightarrow{i} + \mathcal{N}\overrightarrow{j}$ where the number $\mathcal{T}$ can be either positive or negative and the number $\mathcal{N}$ is positive. If $\mathcal{N}$ is constant, then the Coulomb law for the force exerted by the ground can be written under the form:

$$\mathcal{T} \in -\alpha_G \sigma \left(\dot{x} + R\dot{\theta}\right), \tag{1}$$

where $\alpha_G > 0$ is a constant and the multivalued maximal monotone operator σ is defined by:

$$\sigma(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0. \end{cases} \tag{2}$$

Like in [6, p. 146-150], the braking system is composed of two symmetrical sheets that exert a pressure denoted $\alpha_B \geq 0$ (which depends on time) on a disk fixed on the wheel. If the braking system is fixed to the chassis, with a fixed direction according to the ground, then the angular relative velocity between the wheel and the brake is $\dot{\theta}$ and Coulomb's law for the braking torque $\mathcal{M}_B$ exerted on the wheel is written under the form:

$$\mathcal{M}_B \in -\alpha_B \sigma \left(\dot{\theta}\right). \tag{3}$$

This wheel is also submitted to a motor torque $\mathcal{M}$.

Study of a simple model: wheel-chassis-ground

The model of wheel, described in previous section, can be associated other solids in a very complex manner, as in [3, 5, 6, 7]. Here, we study the simple case where a single wheel is fixed to a chassis with a fixed direction according to the ground. This model is not physically realistic! However, we can assume that we have 2 or 4 identical wheels, with the same forces, which keep the chassis horizontal. The vertical loads are assumed to be constant ($\mathcal{N}$ is then constant). We set

$$x_1 = \dot{x}, \quad x_2 = R\dot{\theta}, \quad f = -\mathcal{T}, \quad g = -\mathcal{M}_B. \tag{4}$$

So, the dynamics of this model is given by the following inclusion: m denotes the wheel+chassis mass; R is the wheel's radius, and I its (plane) moment of inertia relative to its center. The two functions $\mathcal{M}$ and α_B are given, and we seek x_1, x_2, f and g satisfying

$$\left\{ \begin{array}{l} m\dot{x}_1 + f = 0, \\ \frac{I}{R^2}\dot{x}_2 + f + g = \mathcal{M}, \\ f \in \alpha_G \sigma\left(x_1 + x_2\right), \\ g \in \alpha_B \sigma\left(x_2\right), \end{array} \right. \tag{5a}$$

with initial conditions

$$\left\{ \begin{array}{l} x_1(0) = x_{1,0} = \dot{x}(0), \\ x_2(0) = x_{2,0} = R\dot{\theta}(0). \end{array} \right. \tag{5b}$$

By considering the state variable $u(t) = \left(\dot{x}(t), R\dot{\theta}(t)\right)$ in $\mathbb{R}^2$, a smooth function $\mathcal{G}$ from $[0, T]$ to $\mathbb{R}^2$, a diagonal matrix with non negative coefficients D and a multivalued operator A_t (which depends on time) from $\mathbb{R}^2$ to $\mathbb{R}^2$, we seet that (5) is equivalent to the following differential inclusion:

$$\dot{u}(t) + DA_t(u(t)) \ni \mathcal{G}(t), \quad \text{a.e. on } (0, T), \tag{6}$$

with initial condition: $u(0) = u_0$. For a fixed value of t, A_t is the subdifferential of the continuous convex function ϕ_t defined by

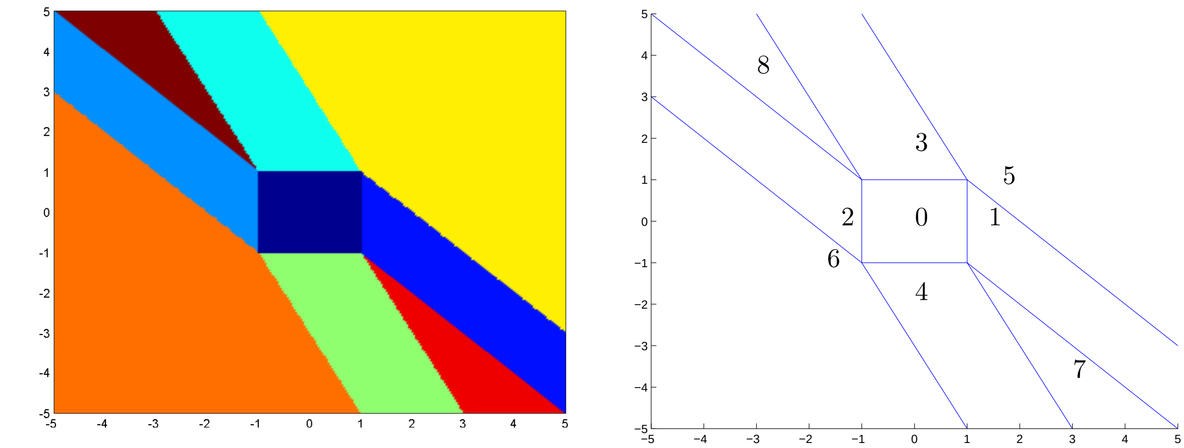
$$\forall (x_1, x_2) \in \mathbb{R}^2, \quad \phi_t(x_1, x_2) = \alpha_G |x_1 + x_2| + \alpha_B(t)|x_2|. \tag{7}$$

If the function α_B is positive and belongs to $L^1(0, T)$ and if $\mathcal{M}$ belongs to $L^1(0, T)$, then there exists a unique solution $u \in W^{1,1}(0, T; \mathbb{R}^2)$ of (6) [26, Theorem 10.5]: we consider $A = \partial\phi$ where $\phi(t, x) = \sum_i \alpha_i(t)\phi(x)$, with ϕ convex and α_i positive.

Numerical simulations

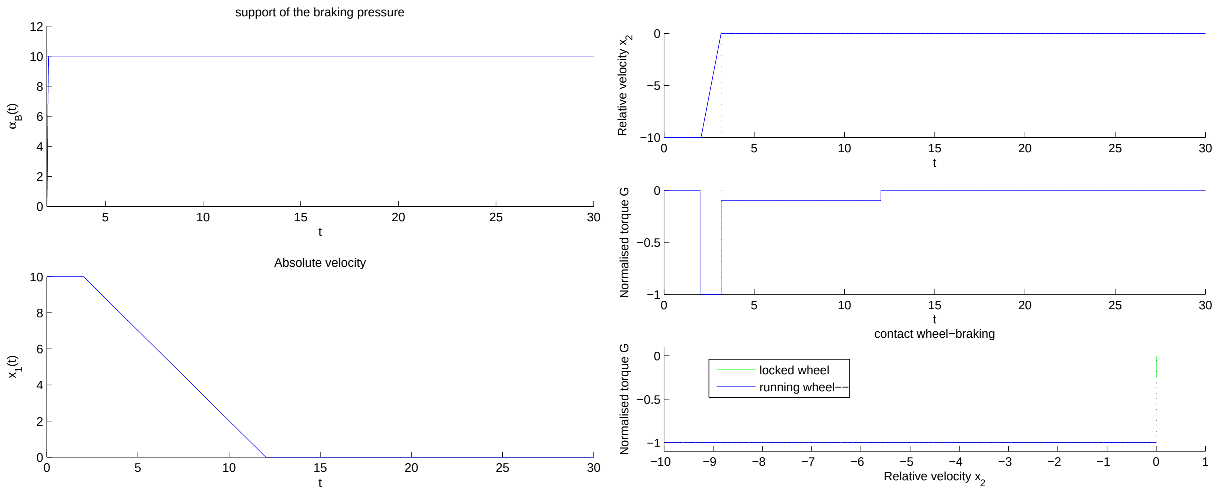
In differential inclusion (6), A depends on time. In this case, results of existence,uniqueness and convergence of [21, 19] remain still valid. We have to determine the resolvent of the maximal monotone operator $\left(I + hDA_{\alpha_B}\right)^{-1}$ where D is diagonal. Then, we obtain explicitly, for all $n \in \{0, \ldots, N-1\}$

$$U^{n+1} = \left(I + hDA_{\alpha_B(t_{n+1})}\right)^{-1} \left(h\mathcal{G}(t_n) + U^n\right). \tag{8}$$



(a) Numerical results (b) Region numbers and theoretical boundaries

Figure 2: Numbers of the nine regions of $\mathbb{R}^2$: a different color is associated with each region number



(a) Support of braking pressure and absolute velocity versus the time (b) Curves of the normalised torque G and of the relative velocity between the wheel and the chassis

Figure 3: Numerical simulations

We define two numbers called $\varepsilon_G \in \{0, 1, -1\}$ and $\varepsilon_B \in \{0, 1, -1\}$ which characterize respectively the kind of the state between the wheel and the ground and between the wheel and the brake.
We may wish to determine it by minimizing non-differentiable convex functions, as explained in [27, exemple 2.3.4] or [23, 24, Theorem 2.9 or section 2.5.6.2]. Here, the convex functions include absolute values, and we will see that the search for minima reveals simple regions of the plane consisting of intersections of half-planes and particular values of ε_G and ε_B. In Fig. 2, we can see nine regions of $\mathbb{R}^2$, each of them being defined by a different color. The central region numbered 0 corresponds to case where the numbers ε_G and ε_B are equal to zero. The eight other regions numbered 1 to 8 correspond to cases where ε_G or ε_B is equal to ± 1.
We can adapt the results of [19, 23, 24], to build an *ad hoc* numerical scheme for the differential problem (6). The convergence error is of order 1. Some numerical simulations shows different behaviors of the wheel.
For example, we consider the following case: the braking torque varies greatly from zero to a maximum value and then decreases back to zero (Fig. 3(a)). On Fig. 3(b), we plotted the normalized torque $G \in [-1, 1]$ (defined by g/α_B, if $\alpha_B \neq 0$ and null otherwise). The relative velocity between the wheel and the chassis is first non equal to zero, with a normalized equal to -1. Then this velocity becomes null and, while the absolute velocity of the wheel is non equal to zero, the normalized torque belongs to $]-1, 0]$.

Different possible generalizations

Example of a “two wheels-chassis-ground” model

Other more complete models can be built by associating one or more wheels with some mechanical components. For example the association of two (or four) wheels from above with chassis (see Fig. 1(b)) is still described by a differential inclusion of the form (6) (in $\mathbb{R}^3$).

More precisely, we now combine the elementary model of the wheel from previous section with an assembly in the following way (see Fig. 1(b)): we consider a set of two wheels (or of four wheels, each identical and subjected to the same forces), with centers O_i, masses m_i and moments of inertia I_i, where $i \in \{1,2\}$, and with a common radius R. We assume that the wheel is in contact with the ground, and denote the action of the ground on wheel i by $\overrightarrow{\mathcal{R}_i} = \overrightarrow{\mathcal{T}_i} + \overrightarrow{\mathcal{N}_i}$. Each of the two wheels supports the chassis, of mass M. We denote by x the abscissa of one point of the chassis and, as previously, we label the rotation of the two wheels using the angles θ_i, where $i \in \{1,2\}$. Let G be the center of mass of the chassis. We write L for the length O_1O_2, and λL and $(1-\lambda)L$, with $\lambda \in]0,1[$, for the distances between O_1 and the projection of G onto (O_1O_2), and between O_2 and the projection of G onto (O_1O_2). We will assume that a braking torque $\mathcal{M}_{B_i}$ (*via* a known pressure α_{B_i}) and a motor torque $\mathcal{M}_i$ are applied to the wheels, with $i \in \{1,2\}$, by two appropriate members which are fixed with respect to the chassis. Each wheel is also submitted to a motor torque $\mathcal{M}_i$.
Thus we have the following different equations: the friction laws are now

$$\forall i \in \{1,2\}, \quad \mathcal{M}_{B_i} \in -k_i \alpha_{B_i}(t) \sigma\left(\dot{\theta}_i\right), \tag{9a}$$

$$\forall i \in \{1,2\}, \quad \mathcal{T}_i \in -\mathcal{N}_i \mu_{f_i} \sigma\left(\dot{x} + R\dot{\theta}_i\right), \tag{9b}$$

where $\mu_{f_i} > 0$ is a constant and the multivalued maximal monotone graph σ is defined by (2).

$$\forall i \in \{1,2\}, \quad \frac{I_i}{R}\ddot{\theta}_i - \mathcal{T}_i - \frac{\mathcal{M}_{B_i}}{R} = \mathcal{M}_i, \tag{9c}$$

$$(m_1 + m_2 + M)\ddot{x} - \mathcal{T}_1 - \mathcal{T}_2 = 0. \tag{9d}$$

We denote by a the ordinate of center of mass the of chassis according to the centers of the wheels and L the distance between the two axis of the wheels. Finally, equating the torque with the time-derivative of the angular momentum twice (in the static case, and neglecting the aerodynamic action in the torque) and applying this at the points O_1 and O_2, along with the vertical equilibrium of both wheels, gives

$$\mathcal{N}_1 = g(m_2 + (1-\lambda)M) - \frac{aM\ddot{x}}{L} - \frac{1}{L}\left(\mathcal{M}_{B_1} + \mathcal{M}_{B_2} + \mathcal{M}_1 + \mathcal{M}_2\right), \tag{9e}$$

$$\mathcal{N}_2 = g(m_1 + \lambda M) + \frac{aM\ddot{x}}{L} + \frac{1}{L}\left(\mathcal{M}_{B_1} + \mathcal{M}_{B_2} + \mathcal{M}_1 + \mathcal{M}_2\right). \tag{9f}$$

As in previous section, we set

$$x_1 = \dot{x}, \quad x_2 = R\dot{\theta}_1, \quad x_3 = R\dot{\theta}_2, \quad , \tag{10a}$$

$$\forall i \in \{1,2\}, \quad f_i = -\mathcal{T}_i, \quad g_i = -\frac{\mathcal{M}_{B_i}}{R}. \tag{10b}$$

If we assume

$$a \ll L \text{ and } \ddot{x} \text{ not too large} \tag{11}$$

(realistic assumption for a car for example), we can neglect $aM\ddot{x}/L$ then we obtain

$$(m_1 + m_2 + M)\dot{x}_1 + f_1 + f_2 = 0, \tag{12a}$$

$$\frac{I_1}{R^2}\dot{x}_2 + f_1 + g_1 = \mathcal{M}_1, \tag{12b}$$

$$\frac{I_2}{R^2}\dot{x}_3 + f_2 + g_2 = \mathcal{M}_2, \tag{12c}$$

$$\mathcal{N}_1 = g(m_2 + (1-\lambda)M) - \frac{1}{L}\left(\mathcal{M}_1 + \mathcal{M}_2\right) + \frac{R}{L}\left(g_1 + g_2\right), \tag{12d}$$

$$\mathcal{N}_2 = g(m_1 + \lambda M) + \frac{1}{L}\left(\mathcal{M}_1 + \mathcal{M}_2\right) - \frac{R}{L}\left(g_1 + g_2\right), \tag{12e}$$

$$f_1 \in \mathcal{N}_1 \mu_{f_1} \sigma\left(x_1 + x_2\right), \tag{12f}$$

$$f_2 \in \mathcal{N}_2 \mu_{f_2} \sigma\left(x_1 + x_3\right), \tag{12g}$$

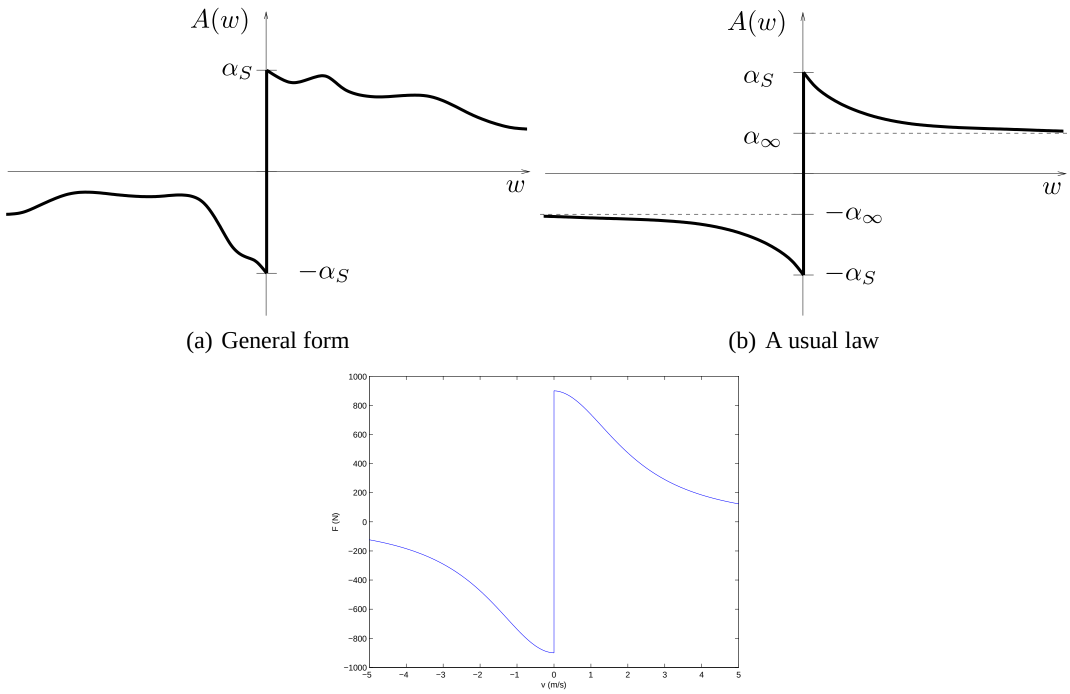
$$g_1 \in \alpha_{B_1} \sigma\left(x_2\right), \tag{12h}$$

$$g_2 \in \alpha_{B_2} \sigma\left(x_3\right). \tag{12i}$$

These equations are valid as long as

$$\mathcal{N}_1 \geq 0, \tag{12j}$$

$$\mathcal{N}_2 \geq 0. \tag{12k}$$



(a) General form (b) A usual law

(c) Stribeck effects

Figure 4: Some examples of friction laws

By using [26, Theorem 9.2], we have existence of solution of (12). Contrary on results on previous section, we have not uniqueness of solutions of (12), as long as the normal forces $\mathcal{N}_i$ depend on solutions *via* $\mathcal{M}_i$ in (12d)-(12e). However, if we assume that

the friction torques $-g_1$ and $-g_2$ appearing in (12d) and (12e) are negligible compared to the torque of the chassis weight (13)

–which is mechanically realistic– we can thus replace (12d) and (12e) by

$$\mathcal{N}_1 = g(m_2 + (1-\lambda)M) - \frac{1}{L}\left(\mathcal{M}_1 + \mathcal{M}_2\right), \tag{14a}$$

$$\mathcal{N}_2 = g(m_1 + \lambda M) + \frac{1}{L}\left(\mathcal{M}_1 + \mathcal{M}_2\right). \tag{14b}$$

In this case, equations (12) can be rewritten under the form (6) (in $\mathbb{R}^3$). As previously, uniqueness and convergence of numerical scheme are then obtained.

More complete mechanical models

The different dynamics of the different constituents of a vehicle may be considered to be longitudinal, transverse and vertical, while taking into account in a more systematic way the different non-linear sources (damper, tyre), and also considering the different transmission devices of the braking or motor torque. In every case, we put the equations governing the motion into the form $u'(t) \in \mathcal{F}(t, u(t))$ where $\mathcal{F}$ is a multivalued mapping from $[0, T] \times \mathbb{R}^p$ to the set of parts $\mathcal{P}\left(\mathbb{R}^p\right)$ of $\mathbb{R}^p$.

Other friction laws

Like in [28] or [23, 24, chapter 7], we could also replace the simpler Coulomb friction laws by more realistic frictions law (see for example [6, 5, 3]). Indeed, this is just a matter of adding a nonlinear, continuous term to operator σ: (1)-(3) are replaced by $\mathcal{T} \in -\alpha_G A\left(\dot{x} + R\dot{\theta}\right)$ or $\mathcal{M}_B \in -\alpha_B A\left(\dot{\theta}\right)$ where $A = \alpha_S \sigma + \Psi$, where α_S is non negative and Ψ is smooth enough (see Fig. 4).

Conclusion and perspectives

1. Multivalued description for simple wheel or associated to chassis providing classical dynamics vehicle problems [6, 5, 3].
2. Huge advantage of numerical *ad hoc* scheme: low order of convergence (1) but robust and *non event-driven*.
3. Encountered difficulties in a similar context [10, 11, 12, 13, 14, 15, 16, 17] avoided here by using this scheme!

The perspective are the following;

1. Allow unilateral contact to take into account motor or braking torques with high variations or for light vehicules (bike, moto ...) by using work of P. Ballard [29] and without assumption (12j)-(12k) ;
2. Treat non uniqueness problems and/or avoid assumption (13) by using for example works [29, 30] ;
3. Associate this scheme with high order scheme (on intervals without change of state).

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