A driven and braked wheel described by differential inclusion – Applications

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Content

Abstract

A wheel subjected to two friction forces, one exerted by the ground and one exerted by the brake pad, is studied. A formalism using multivalued operators allows us to write the constitutive laws of this wheel in the form of a differential inclusion, for which we have the existence and the uniqueness of the solution, as well as the convergence of the associated numerical scheme. This differential inclusion takes into account all the possible cases in a same equation (slip or stick phases).

This wheel may be connected by itself to a chassis. The chassis may also be connected to two or four of these wheels. In these two cases, we again obtain a well-posed differential inclusion, in terms of existence, uniqueness and convergence of the numerical scheme.

In a more general manner, numerous applications can be proposed in the domain of the non-linear dynamics of wheeled vehicles.

This formalism allows to write correctly the non-linear dynamics of wheeled vehicles with well posed differential inclusions.

These results come out from [Bas13b; Bas14].

Content

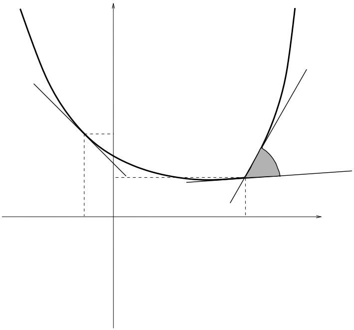
Sub-differential of convex functions from $\mathbb{R}$ to $\mathbb{R} \cup \{+\infty\}$

Definition

If ϕ is porper convex, the sub-differential $\partial\phi$ is defined by

$$y \in \partial\phi(x) \iff \forall\xi \in \mathbb{R}, \quad \phi(\xi) \geq y(\xi - x) + \phi(x). \qquad (1)$$

The inequation (1) means that y is the slope of a line d. The point $(x, \phi(x))$ belongs to d and all the line is under the graph of ϕ.



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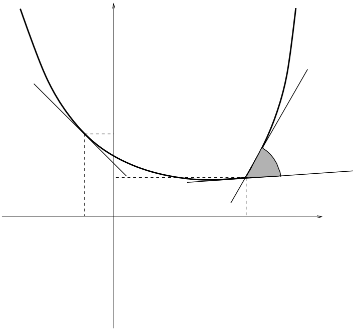
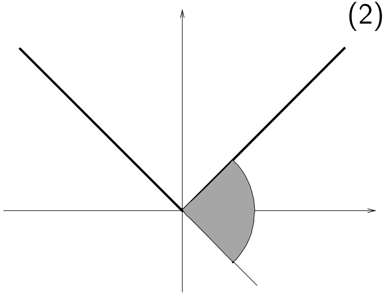


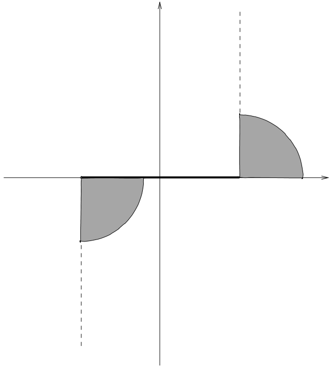
Sub-differential of convex functions from $\mathbb{R}$ to $\mathbb{R} \cup \{+\infty\}$

Examples of graphes σ and β

$$\sigma(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0. \end{cases} \tag{2}$$



$$\beta(x) = \begin{cases} \mathbb{R}_- & \text{if } x = -1, \\ \mathbb{R}_+ & \text{if } x = 1, \\ 0 & \text{if } x \in (-1, 1), \\ \emptyset & \text{if } x \notin [-1, 1]. \end{cases} \tag{3}$$



Sub-differential of convex functions from $\mathbb{R}$ to $\mathbb{R} \cup \{+\infty\}$

Maximal monotone properties

- The sub-differential of convex functions are maximal monotone.
- If the multivalued operator A is maximal monotone, for all $\lambda > 0$

$$\forall y \in \mathbb{R}, \quad \exists! x \in \mathbb{R}, \quad x + \lambda A x \ni y. \tag{4}$$

- We consider then the resolvent $(I + \lambda A)^{-1}$ of A.

Extension to any Hilbert space H

- All previous definitions can be extented if we replace the product of $\mathbb{R}$ by a scalar product.
- If ϕ from H to $\mathbb{R} \cup \{+\infty\}$ is convex and differentiable in x, we have

$$\partial\phi(x) = d\phi(x). \tag{5}$$

- In particular, if H is of finite dimension,

$$\partial\phi(x) = \nabla\phi(x). \tag{6}$$

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Existence, uniqueness and numerical scheme

- We assume that A is maximal monotone on H and f is regular. According to [Bré73], we have existence and uniqueness of the following differential inclusion

$$\dot{u}(t) + A(u(t)) \ni f(t, u(t)), \tag{7}$$

with initial conditions.

- In all cases, we have $A = \partial\phi$.
- The (semi) implicite numerical scheme is

$$\frac{U^{n+1} - U^n}{h} + A(U^{n+1}) \ni f(t_n, U^n), \tag{8}$$

equivalent to

$$U^{n+1} = (I + hA)^{-1} \left(hf(t_n, U^n) + U^n\right). \tag{9}$$

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Existence and uniqueness results

Theorem ([BS02])

Let H be a separable Hilbert space, A a maximal montone, $u_0 \in D(A)$ and $f : [0, T] \times H \to H$ such that

$$\exists L, \quad \forall (t, x_1, x_2) \quad \| f(t, x_1) - f(t, x_2) \| \leq L \| x_1 - x_2 \| , \qquad (10)$$

and for all $R \geq 0$,

$$\Phi(R) = \sup \left\{ \left\| \frac{\partial f}{\partial t}(., v) \right\|_{L^2(0,T;H)} : \|v\|_{L^2(0,T;H)} \leq R \right\} < +\infty.$$

Then, there exists a unique fonction $u \in W^{1,\infty}(0, T; H)$ of (7), with $u(0) = u_0$.

J. Bastien and M. Schatzman. "Numerical precision for differential inclusions with uniqueness". In: *M2AN Math. Model. Numer. Anal.* 36.3 (2002), pages 427–460. DOI: 10.1051/m2an:2002020

Convergence of numerical scheme

Theorem ([Bas13a])

Let H be a separable Hilbert space, ϕ a porper convexe, l.s.c (lower semi-continuous), $u_0 \in D(\partial\phi)$ and $f : [0,T] \times H \to H$ verifying (10) *and, for all $R \geq 0$,*

$$\Phi(R) = \sup\left\{ \left\| \frac{\partial f}{\partial t}(.,v) \right\|_{L^\infty(0,T;H)} : \|v\|_{L^2(0,T;H)} \leq R \right\} < +\infty.$$

Let u be the unique solution of (7) *with $A = \partial\phi$ and u_h defined by the numerical scheme* (9)*. There exists a constant M such that for all h, we have*

$$\forall t \in [0,T], \quad \|u_h(t) - u(t)\| \leq Mh.$$

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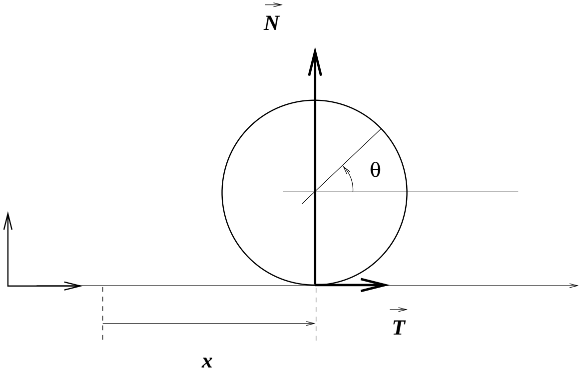
Applications

Numerous models: see [BSL00; BSL02; BL05; LBB05; BL08; BL09; BBL12; BBL13; Lam+09; Lam+12]

Content

Definition of studied solid



2 degrees of freedom:

- Abscissa x;
- Angle θ.

- Force exerted by ground on the wheel $\vec{\mathcal{T}} + \vec{\mathcal{N}}$;
- Braking torque exerted on the wheel $\mathcal{M}_B$;

Friction constitutive law's

For ground force

$$\mathcal{T} \in -\alpha_G \sigma \left(\dot{x} + R\dot{\theta} \right), \tag{11}$$

where $\alpha_G > 0$ is a constant and the multivalued maximal monotone operator σ is defined by (2) and for braking torque

$$\mathcal{M}_B \in -\alpha_B \sigma \left(\dot{\theta} \right). \tag{12}$$

This wheel is also submitted to a motor torque $\mathcal{M}$.

Dynamics

We write

$$x_1 = \dot{x}, \quad x_2 = R\dot{\theta}. \tag{13}$$

m denotes the wheel+chassis mass. If we set

$$f = -\mathcal{T}, \quad g = -\mathcal{M}_B, \tag{14}$$

then the problem is equivalent to the following: the two functions $\mathcal{M}$ and α_B are given, and we seek x_1, x_2, f and g satisfying

$$\begin{cases} m\dot{x}_1 + f = 0, \\ \frac{I}{R^2}\dot{x}_2 + f + g = \mathcal{M}, \\ f \in \alpha_G \sigma\left(x_1 + x_2\right), \\ g \in \alpha_B \sigma\left(x_2\right). \end{cases} \tag{15}$$

This formulation is the multivalued version of equations used for example in [TEKB14; Bro06; Bro09].

Existence and uniqueness

Definition

Let $\alpha \geq 0$ be a real number. The multivalued operator A_α is defined from $\mathbb{R}^2$ to $\mathbb{R}^2$ and, to each $X = (x_1, x_2)$, associates $Y = (y_1, y_2)$ defined as follows. There exist f and g in $\mathbb{R}$ such that

$$f \in \alpha_G \sigma(x_1 + x_2), \tag{16a}$$

$$g \in \alpha\sigma(x_2), \tag{16b}$$

$$y_1 = f, \tag{16c}$$

$$y_2 = f + g. \tag{16d}$$

Existence and uniqueness

Lemma

For all $\alpha \geq 0$, A_α is maximal monotone. Moreover, it is equal to the subdifferential of the convex function ϕ defined by

$$\forall (x_1, x_2) \in \mathbb{R}^2, \quad \phi(x_1, x_2) = \alpha_G |x_1 + x_2| + \alpha |x_2|. \tag{17}$$

Proof.

Formally, by using (6) and then $\sigma = \partial|.| = |.|'$, we write

$$\partial\phi(x_1, x_2) = \begin{pmatrix} \alpha_G \frac{\partial}{\partial x_1} |x_1 + x_2| \\ \alpha_G \frac{\partial}{\partial x_2} |x_1 + x_2| + \alpha \frac{\partial}{\partial x_2} |x_2| \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_G \sigma(x_1 + x_2) \\ \alpha_G \sigma(x_1 + x_2) + \alpha\sigma(x_2) \end{pmatrix} = \begin{pmatrix} f \\ f + g \end{pmatrix} = A_\alpha(x_1, x_2)$$

Rigorously, see [Bas13b; Bas14]. □

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&= \begin{pmatrix} \alpha_G \sigma(x_1 + x_2) \\ \alpha_G \sigma(x_1 + x_2) + \alpha\sigma(x_2) \end{pmatrix} = \begin{pmatrix} f \\ f + g \end{pmatrix} = A_\alpha(x_1, x_2)
\end{aligned}$$

Rigorously, see [Bas13b; Bas14]. □

Differential inclusion and numerical scheme

- Differential inclusion (15) is of the form (7), but A depends on time. We apply existence and uniqueness results of [Dei92] with $A = \partial\phi$ where $\phi(t,x) = \sum_i \alpha_i(t)\phi(x)$, with ϕ convex and α_i positive.
- We have to determine the resolvent of the maximal monotone operator $(I + hDA_{\alpha_B})^{-1}$ where D is diagonal.
- Then, we obtain explicitly, for all $n \in \{0, \dots, N-1\}$

$$U^{n+1} = \left(I + hDA_{\alpha_B(t_{n+1})}\right)^{-1}\left(h\mathcal{G}(t_n) + U^n\right). \qquad (18)$$

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Calculus of $(I + hDA_{\alpha_B})^{-1}$

- We define two numbers called $\varepsilon_G \in \{0, 1, -1\}$ and $\varepsilon_B \in \{0, 1, -1\}$ which characterize respectively the kind of the state between the wheel and the ground and between the wheel and the brake.
- We may wish to determine it by minimizing non-differentiable convex functions, as explained in [Bré73, exemple 2.3.4] or [BBL12; BBL13, Theorem 2.9 or section 2.5.6.2]. Here, the convex functions include absolute values, and we will see that the search for minima reveals simple regions of the plane consisting of intersections of half-planes and particular values of ε_G and ε_B.

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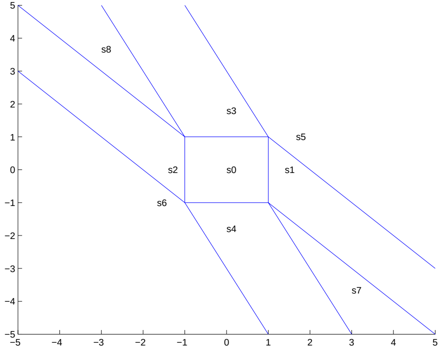


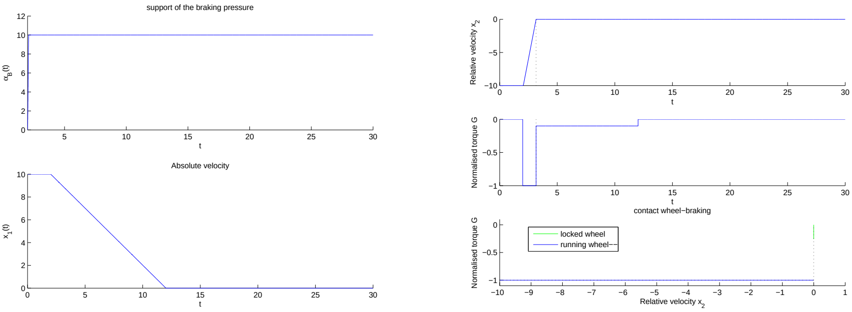
In this figure, we can see nine regions of $\mathbb{R}^2$, each of them being defined by a different color. The central region numbered 0 corresponds to case where the numbers ε_G and ε_B are equal to zero. The eight other regions numbered 1 to 8 correspond to cases where ε_G or ε_B is equal to ± 1.

Numerical simulations

We conduct three numerical simulations:

1. The first one is a sport drive: the motor torque varies greatly from zero to a maximum value and then decreases back to zero.
2. The second one corresponds to a sport braking: the braking torque varies greatly from zero to a maximum value and then decreases back to zero.
3. The third one is a smooth ride: the motor torque and braking torque vary slowly.

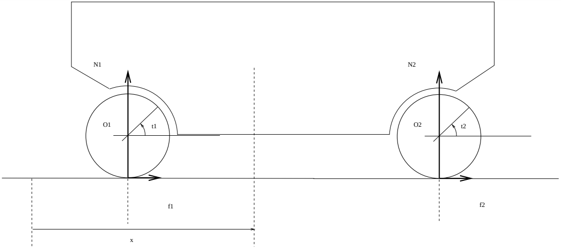
Numerical simulations



For the case 2, we obtained the normalized torque $G \in [-1, 1]$ (defined by g/α_B, if $\alpha_B \neq 0$ and null otherwise) on above figure. The relative velocity between the wheel and the chassis is first non equal to zero, with a normalized equal to -1. Then this velocity becomes null and, while the absolute velocity of the wheel is non equal to zero, the normalized torque belongs to $]-1, 0]$.

Content

Generalization



Other more complete models can be built by associating one or more wheels with some mechanical components. For example the association of two (or four) wheels from above with chassis is still described by a differential inclusion of the form (15) (in $\mathbb{R}^3$). We obtain then a theoretical framework that provides the existence and uniqueness of the solution and convergence of numerical scheme. Like in [BBL12; BBL13, chapter 7], we could also replace the simpler Coulomb friction laws by more realistic frictions law (see for example [Bro09; Bro06; AA05]).

Content

Conclusion

- Multivalued description for simple wheel or associated to chassis providing classical dynamics vehicle problems [Bro09; Bro06; AA05].
- Huge advantage of numerical *ad hoc* scheme: low order of convergence (1) but robust and *non event-driven*.
- Encountered difficulties in a similar context [Hof06; Hof08; Sch09; TA02; TT10; TEKB14; Xia02; XT04] avoided here by using this scheme!

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Perspectives

1. Allow unilateral contact to take into account motor or braking torques with high variations or for light vehicules (bike, moto ...) by using work of P. Ballard [BB05] ;
2. Treat non uniqueness problems by using for example works [BB05; CB14] ;
3. Associate this scheme with high order scheme (on intervals without change of state).

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